Integrate Science and Mathematics for Secondary Education

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> Pearson, Moje, and Greenleaf: *Scientific literacy has been the rallying cry of science education reform for the past 20 years …*
>
> (Pearson, Moje, & Greenleaf, 2010, p. 459)

> O'Halloran: *The discourse is presented in such a way that many students do not understand what mathematics is, or how mathematical symbolism developed historically as a semiotic resource in order to fulfill certain functions.*
>
> (O'Halloran, 2005, p. 202)

Abstract

Compelling reasons to integrate mathematics and science education include the following. Science and mathematics are indelibly interlinked through their co-evolution and resulting symbiotic functionality. Science grounds mathematics both in meaning and in value. Mathematics provides science with the language of patterns, associated concepts, and extension of reasoning. Together mathematics and science combine to provide and make explicit conceptualization that far exceeds that lacking either, precisely the barrier to mathematical and scientific literacy. Integration of mathematics and science education must target and exploit these inherent synergies to help students build an internalized dictionary of symbolic forms that is established through connection to real world entities and through use to achieve a real world project. In this context, science and mathematics are tools for deep understanding as well as effecting action in the world. This showcases and helps students internalize science and mathematics in their proper context, demonstrating the inherent value and purpose of science and mathematics that is often missing, and makes such a difference in fostering attitudes that can further the trajectories of future studies and careers. A set of early middle school lessons for a scale model building project are outlined and discussed to address and make explicit many of these ideas. A scale model building project provides one way to integrate mathematics and science education in a way that meets these ambitious goals.

Benefits of Integrating Mathematics and Science Education

The most crucial benefit of integrating mathematics and science education is also a major goal of both mathematics and science education. That goal is to build a foundation on which to develop students' ability to both understand and to use the methods and components of both science and mathematics to gain deeper qualitative and quantitative understanding in order to better make decisions about and take action in the world.

At the heart of the matter, and the issue which is the most troublesome for education, is the extension beyond words alone into the multifaceted realm of deep understanding where words, symbols, and diagrams merge to yield new qualitative and quantitative meaning. This inherent synergy of science and mathematics forms the basis for the scientific world view that traces its roots back to the epitome of the scientific revolution where Newton used Decartes' landmark development of symbolic mathematics to create a workable model of classical mechanics (how and why things move).

This quintessential example provides a guide for identifying key components that yield the full meaning and value of integrating math and science:

- extended understanding through mapping mathematical models to real world phenomena,
- creation of concepts and relations through a combination of diagrams, symbols, and language,
- extended reasoning through a combination of diagrams, symbols, and language,
- creation of a quantitative, predictive model that unifies previously disparate phenomena,
- comprehension of the works of others both through writings and conversation,
- written exposition of the unifying model.

Implicit in this is the natural use of mathematics in context that serves to:
- motivate one to learn and use mathematics by placing it in one of its most valuable and enjoyable settings (Einstein's quote about "difficulties in Mathematics" notwithstanding),
- provide understanding and unification of phenomena that surpasses others (including simulations) through the use of explicit visual statements in the language of patterns,
- give meaning and value to otherwise abstract pattern manipulation.

These are basic components of mathematical and scientific literacy that are often missing in secondary education, difficult to acquire in short time, yet are fundamental to STEM studies and careers. The pages that follow investigate these precepts in depth to make explicit not only key aspects of how they apply to gaining scientific and mathematical literacy, but also to conclude that an integrated setting not only fosters the natural motivations of both learning mathematics and using it to understand and unify scientific phenomena, but also develops lasting value and favorable perspectives that foster deeper and further learning and use.

Science and Mathematics are Fundamentally Linked

At the highest levels of literacy, mathematics makes explicit both quantitatively and qualitatively the underlying patterns of science, science gives grounded meaning to components of the patterns, and the combination creates our resulting perceptions of reality (Pearson, Moje, & Greenleaf, 2010, p.459) (Wellington & Osbourne, 2001, pp.138-140). O'Halloran (2005) explained the depths to which representation and meaning making link mathematics and science:

> The new approach advocated by Decartes proved to be significant because Newton and others created a movement which involved a new representation of the physical world using new semiotic tools. In this movement, matter and perceptual data were re-admitted by Newton, but in a new mathematicized form … (p. 57)
>
> Newton's new semiotic constructions explained the visible world through invisible properties which were made 'real' or 'concrete' through mathematical symbolic description. *One key to this success was that the mathematical symbolism, the visual images, and language worked together* [emphasis added]. (p. 57)

Sherin (2001) emphasized this linkage between meaning making in both mathematics (symbolic) and science (conceptual) at the detailed level of symbolic forms in the following way: "In contrast, the schemata associated with *symbolic forms* [emphasis added] are conceptual schemata, in which the *structure corresponds to an understanding of conceptual relations or structure in the world* [emphasis added]." (p. 497)

Common Core State Standards for Mathematics acknowledged the essential nature of this link in the introduction of high school standards by stressing the importance of modeling via mathematics both as a category that cuts across all other conceptual categories and as a standard mathematical practice (National Governors Association Center & Council of Chief State Schools Officers, 2010, p. 57). The

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Wellington, J., and Osborne, J. (2001). *Language and literacy in science education*. Philadelphia, PA: Open University Press.

Next Generation Science Standards (NGSS) acknowledged in slightly different form in its first paragraph for middle school through the expectations that students develop "usable knowledge to explain real world phenomena" through scientific practices that include "using models, … , using mathematical and computational thinking, and constructing explanations." with similar expectations for high school students, replacing "real world phenomena" with "ideas" (Krajcik, 2013, pp. 47, 75).

Redish and Kuo (2015) expressed the problem when the link fails as it commonly does for high school through college (graduate level) physics students:

> A few students seem to take naturally and comfortably to the use of math to describe the physical world, but many struggle with it, both at the introductory and at the more advanced level, even though multiple math classes may be required as prerequisites for the physics classes. (p. 562)

Sherin (2001) traced this symptom to the root cause with a detailed theory of symbolic forms we'll examine shortly, but also gave a higher level summary:

> We now believe that it is not enough for students to be able to solve problems or answer certain questions; in addition we want students to have something called *understanding that lies behind and undergirds such abilities* [emphasis added]. In domains in which formal mathematical expressions are prevalent, these issues come into particularly sharp focus. We do not want meaningless symbol manipulation; if students use symbolic expressions, we want them to *use symbols with understanding* [emphasis added]. (p. 479)

The most crucial inherent synergy between mathematics and science is this very linkage between the two and what most needs to be demonstrated, experienced, and developed within students throughout secondary education. The following sections look more closely at the components of this link to make explicit their key characteristics, associated value, and necessity as targets for integrated science and mathematics education.

Science Grounds Mathematics

One of the most important benefits of integrating mathematics and science education is that using mathematics in the physical context of observation-based science grounds the meaning of abstract symbolism in a fundamentally perceptual way. In other words, it connects abstract symbols to personal sensations (visions, sounds, feeling) that gives the most fundamental and internalized meaning: the ground floor of meaning, our sensations. Indeed, Redish and Kuo (2015, pp. 569-570) referenced Lakoff and Nuñez (2001), and Füster (1999) to explain that grounding infuses meaning into symbolic thinking common to both mathematics and science:

> … ultimately our conceptual system is grounded in our interaction with the physical world: How we construe even highly abstract meaning is constrained by and is often derived from our very concrete experiences in the physical world. … The idea is that (a) our close sensorimotor interactions with the external world strongly influence the structure and development of higher cognitive facilities, and (b) the cognitive routines involved in performing basic physical actions are involved in even higher-order abstract reasoning (Füster, 1999).
>
> The grounding of conceptualization in physical experience and actions also extends to higher cognitive processes such as mathematical reasoning. … Although formal mathematics may, at times, seems distinct from everyday life, one's cognition about and understanding of such formal rules may be grounded in common, physical experiences. (pp. 569-570)

Goldstone, Landy, and Son (2008) emphasized the perception aspect of this last observation in the interpretation of results of their study focused on grounded cognition and formation of transportable knowledge and skills:

> A second series shows that even when students are solving formal algebra problems, they are greatly influenced by non-symbolic perceptual grouping factors. We interpret both results as showing that high-level cognition that might seem to involve purely symbolic reasoning is actually driven by perceptual processes. The educational implication is that instruction in science and mathematics should involve not only the teaching of abstract rules and equations but also training students to perceive and interact with the world. (p. 2)

Sherin (2001) expressed similar sentiments in literacy terms based on the concept of symbolic forms that associate symbolic templates and conceptual meaning as entries in a sort of crossed indexed dictionary facilitating perception and meaning:

> … successful physics students learn to express a moderately large vocabulary of simple ideas in equations and to read these same ideas out of equations. I call the elements of this vocabulary *symbolic forms*. Each symbolic form associates a simple conceptual schema with an arrangement of symbols in an equation. Because they possess these symbolic forms, students can take a conceptual understanding of some physical situation and express that understanding in an equation. Furthermore they can look at an equation and understand it as a particular description of a physical system. (p. 482)

Sherin (2001) goes on to highlight at various points the importance of connecting symbolism to specific contexts and the subtle effect it has on activating symbolic forms via cues, both functional and situational, as is necessary to both understand symbolic patterns at an abstract level and simultaneously use them to deeper understand a new context:

> With regard to mechanism, the key question concerns when and how a symbolic form is cued to an active state. I describe only simple mechanisms and then only in broad strokes. First a form can be activated by being recognized in an equation. A person looks at an equation and sees the symbolic form there. Alternatively, an understanding of the physical situation to be described may activate a form. The situation is understood, and then this understanding in some manner activates a particular symbolic form. (p. 504)
>
> Even if students possess many of the necessary symbolic forms, however, getting these elements engaged in the right places may not be a trivial instructional goal. Students must learn to adopt particular stances to individual physics expressions … (p. 522)

Notice that the value of grounding abstract mathematical symbolism is only achieved when mathematics is being ***used in context*** to make explicit and understand underlying patterns: associating the symbolism with perceptually meaningful aspects of the world. An immediate conclusion is that (a) the more fundamentally perceptible the pattern, (b) the more basic the symbolism, and (c) the more meaningful and complete the association, then (d) the more deeply internalized the symbolic form may be embedded, and (e) the more easily activated thereafter. We'll return to the perception and deep embedding aspects of this theme below after a few observations on the meaning and value of the association.

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Scale Model Building Reading 2
Venema (2013) *Exploring Advanced Euclidean Geometry with GeoGebra*
pp. ix.

Proof

A major accomplishment of the ancient Greeks was the introduction of logic and rigor into geometry. They *proved* their theorems from first principles and thus their results are more certain and lasting than are mere observations from empirical data. The logical, deductive aspect of geometry is epitomized in Euclid's *Elements* and proof continues to be one of the hallmarks of geometry to this day.

Until recently, all those who worked on advanced Euclidean geometry followed in Euclid's footsteps and did geometry by proving theorems, using only pencil and paper. Now that computer programs such as GeoGebra are available as tools, we must reexamine the place of proof in geometry. Some might expect the use of dynamic software to displace the deductive approach to geometry, but there is no reason the two approaches cannot enhance each other. I hope this book will demonstrate that proof and computer exploration can coexist comfortably in geometry and that each can support the other.

The exercises in this book will guide the student to use GeoGebra to explore and discover the statements of the theorems and then will go on to use GeoGebra to better understand the proofs of the theorems as well. At the end of this process of discovery the student should be able to write a proof of the result that has been discovered. In this way the student will come to understand the material to a depth that would not be possible if just computer exploration or just pencil and paper proof were used and should come to appreciate the fact that proof is an integral part of exploration, discovery, and understanding in mathematics.

Not only is proof an important part of the process by which we come to discover and understand geometric results, but the proofs also have a subtle beauty of their own. I hope that the experience of writing the proofs will help students to appreciate this aesthetic aspect of the subject as well.

In this text the word "verify" will be used to describe the kind of confirmation that is possible with GeoGebra. Thus to *verify* that the angle sum of a triangle is 180° will mean to use GeoGebra to construct a triangle, measure its three angles, calculate the sum of the measures, and then to observe that GeoGebra reports that the sum is always equal to 180° regardless of how the size and shape of the triangle are changed. On the other hand, to *prove* that the angle sum is 180° will mean to supply a written logical argument based on the axioms and previously proved theorems of Euclidean geometry.

Science Gives Value To Mathematics

Grounding is just one way science adds value to mathematics. From another direction, science gives mathematics value by making it useful: extracting value from the mathematics. This happens in a number of ways. First, using mathematics to make visually explicit underlying patterns of real-world phenomena showcases the values of mathematics in stripping away all but the pattern itself. This allows attention to focus on only the most crucial aspects of a model or explanation, often yielding a deeper or more complete understanding of the phenomena itself. However a second aspect is the use of similar patterns in transferring meaning and understanding from one set of real-world phenomena to another, as described by Lin and Singh (2011) in their study of students of introductory physics:

> Students can be explicitly taught to make an analogy between a solved problem and a new problem, even if the surface features of the problems are different. In doing so, students may develop an important skill shared by experts: the ability to transfer from one context to another, based upon shared deep features. (pp. 1-2)

Here again, the most value is added when the mathematics *fully grounded in one context*, where full identifications of symbols with real world entities and behaviors, relationships, and intuitions, can be transformed to another context less familiar, but with the same underlying patterns, so that meanings and intuitions can be meaningfully transferred to the new realm. This is done routinely in engineering where the more fundamentally perceptible notions of classical mechanics (force, velocity, friction, …) are applied to transfer intuition into the realm of electric circuits (voltage, current, resistance, …). Furthermore, when both contexts have perceptually grounded symbolic forms the opportunity for deeper abstraction, unification, and conceptualization is possible. For instance, as more insight in electrodynamics is developed and combined with analogous mechanics concepts, more abstract and unifying concepts and principles (such as the system energy and minimization principles) naturally emerge. One famous, but increasingly underappreciated, example of the power of this transference is Maxwell's use of a classically mechanical model in combination with electromagnetic phenomena to develop a fully unified theory of electrodynamics that resulted in the landmark conceptualization that light is an electromagnetic wave (Siegel, 1991), a cornerstone of modern physics:

> It was the theory of molecular vortices that was uppermost in Maxwell's mind, and it was by means of that mechanical representation that Maxwell intended to unify electromagnetic theory; a complete and consistent set of equations would be a necessary part of the exercise, but the center of concern was the mechanical picture, rather than the electromagnetic equations considered in and of themselves as disembodied mathematical entities. (pp. 96-97)

In yet another direction, science transfers the value of its resulting beneficial effect in the world to the mathematics it is using. Redmond et al. (2011) reported the motivation generated by students realization of significant beneficial effect to others: "… they see a purpose behind what they're doing. They're trying to do a good job because they know that what they do is going to impact somebody's life … ." As important as the motivation itself are the resulting associated attitude and educational impacts manifested through their 2 year engineering project for middle school students (Redmond et al., 2011):

> This study provides important results related to improving science and mathematics instruction and engineering career awareness with middle level students. Taken together, this data show that the Get a Grip engineering curricula seemed to have a significant impact on all middle school students' (1) confidence in science and mathematics; (2) effort toward mathematics and science; (3) awareness of engineering; and (4) interest in engineering as a potential career. (p. 406)

Once again, the value is added in the use of mathematics to understand and affect the real world, primarily because it is being used in a highly relevant and meaningful context.

Mathematics Provides the Quantitative Language of Patterns

Conversely, mathematics provides to science the language “specifically designed as a semiotic resource to describe patterns which can be rearranged for the solution to problems” (O’Halloran, 2005, p. 118), providing concise symbolic representations that extend not only meanings, but also methods of thinking, both qualitatively and quantitatively. O’Halloran (2005) explained in great detail from which the following synopses give some sense:

> The textual organization of mathematical symbolism is sophisticated and highly formalized in order to facilitate the economical encoding of relations which permits immediate engagement with the experiential and logical meaning … (p. 96)
>
> The symbolism developed a functionality through new grammatical systems which permitted expansions beyond that capable with language, but at the same time it depended upon employing certain linguistic elements and a range of grammatical strategies inherited from language. … (p. 104)
>
> With the development of symbolic algebra, attention turned to generalized descriptions of relations using algebraic methods. The success of these descriptions meant that mathematical symbolism developed as a semiotic resource with grammatical systems which were unique to that resource. These systems developed in accordance with the aim of mathematics: the description of patterns and means to solve problems relating to these descriptions. (p. 121)

The symbols of mathematics provide an exquisitely structured language of relations, operations, and pattern structures designed to make immediately visible and comprehensible patterns and sequences of patterns that extend reasoning and cognition to a new dimension. Again, O’Halloran (2005) explained in detail with examples using a level-of-abstraction (or rank) mechanism called rankshifting, from which the following extract gives some flavor:

> The potential for *rankshifted configurations of Operative processes and participants* is one of the key factors in the success of mathematical symbolism because this strategy *preserves* process/participant structures which may be reconfigured for solution to problems. This is a significant point in understanding how the grammar of mathematical symbolism is functionally organized to fulfill the goal of mathematics: to order, to model situations, to present patterns, to solve problems, and to make predictions. … The degree of rankshifting found in mathematical symbolism exceeds that found in language. … At each stage, the process/participant configuration is preserved so that the expression can be rearranged and simplified. This is an important grammatical strategy in the evolution of mathematical symbolism as the semiotic which is used to solve problems.

This visibly explicit, concise encoding of pattern structure and sequence is crucial, as mentioned above, in order to visibly cache the pattern (and/or sequence of patterns), so as to focus attention to the complete pattern (or sequence), and extend comprehension.

An example from the third lesson of the scale model building project below involves diagrams and the following reasoning symbology to justify the conclusion that triangles at the viewpoint in the photo and the real world object are similar (have the same shape):

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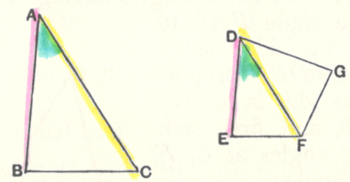
PROPOSITION 6.

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let ABC, DEF be two triangles having one angle BAC equal to one angle EDF and the sides about the equal angles proportional, so that,

as BA is to AC, so is ED to DF;

I say that the triangle ABC is equiangular with the triangle DEF, and will have the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE.



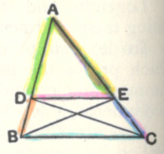
Euclid's Similar Triangle Propositions from Book VI
For the discussion of similar triangles.
Heath (1956) *Euclid: The Thirteen Books of the Elements* (2nd Ed.) (Vol. 2) pp. 194-205.

PROPOSITION 2.

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

For let DE be drawn parallel to BC, one of the sides of the triangle ABC;

I say that, as BD is to DA, so is CE to EA.



PROPOSITION 5.

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

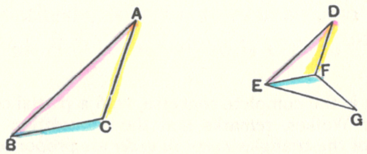
Let ABC, DEF be two triangles having their sides proportional, so that,

as AB is to BC, so is DE to EF,

as BC is to CA, so is EF to FD,

and further, as BA is to AC, so is ED to DF;

I say that the triangle ABC is equiangular with the triangle DEF, and they will have those angles equal which the corresponding sides subtend, namely the angle ABC to the angle DEF, the angle BCA to the angle EFD, and further the angle BAC to the angle EDF.



ΔABD ~ ΔAEG	→	AB : AE :: BD : EG	→	AB/AE = BD/EG.
ΔABC ~ ΔAEF	→	AB : AE :: BC : EF	→	AB/AE = BC/EF.
		BD/EG = AB/AE = BC/EF		and
		angle B = angle E	→	ΔBCD ~ ΔEFG

The point is that the diagrams and symbology are compact enough to be able to hold in attention simultaneously, while content rich enough to represent the necessary technical features. These include various objects, triangles, angles, lengths, ratios, and the reasoning relating it all. Though each piece has meaning individually, only when combined and in the context of real world correspondents and intent, does the full meaning become manifest. Our visual processing systems function autonomously to bring together the technical content of the diagrams, corresponding symbolic relationships, and grounded real-world objects to create deep meaning that surpasses the pieces. This works in much the same way that adding features such as red, round, about the size of a fist, crunchy, sweet, ... eventually become the representation for an apple, even without the name.

We intend for a similar phenomenon to occur for students of high school chemistry, with the combination of internal structure of atoms (protons, neutrons, electrons), geometry of electron orbitals (both in 3D and abstractly in the periodic table), and molecular formulas and redox equations to form surpassing mental visualizations and conceptualizations that make chemistry not only comprehensible, but quantitative and intuitive.

Visualization is the primary conduit to deep meaning for the world beyond our immediate reach: the goal of our visual system is to generate representations of real world objects and automatically attach deeper associated meaning. Arcavi (1999, p. 2) quoted Adams and Victor (1993, p. 207):

> The faculty of vision is our most important source of information about the world. The largest part of the cerebrum is involved in vision and in the visual control of movement, the perception and the elaboration of words, and the form and color of objects. The optic nerve contains over 1 million fibers, compared to 50,000 in the auditory nerve. The study of the visual system has greatly advanced our knowledge of the nervous system. Indeed, we know more about vision than about any other sensory system. (p. 2)

However, rather than the input of the optic nerve, or the large portion of our brains composed of various neural systems and specific processing that builds meaning from the bottom up, for the purposes of perception and meaning of symbolic patterns, we have to go higher. To get at meaning associated with the visual patterns of symbols, we need to focus our attention on the "higher levels of cognition" of "visual imagery, visual working memory and visual consciousness" that "pose a challenge to modern visual neuroscience" (Greenlee & Tse, 2008):

> The representations that our visual system creates of events and objects within a 3D scene are so accurate and produced so quickly, that most of us operate under the false impression that we perceive the world in a veridical way. Perceived qualities such as redness or brightness do not exist in the world; they are creations of our brain. (p. 119)

Similarly with qualities such as 3-ness, additivity, or equality. This higher level of perceptual cognition that automatically tacks on the deeper associated meanings is the level with which we are most concerned in education: the higher level perception of symbols, relations, and corresponding concepts that are the focus of mathematical and scientific thought and the resulting created reality that each of us experiences. This reality is based not only on the neural signals our brain receives, but their combination with our preconceived base of categorizations and networks of meaning, as yielded automatically by our higher levels of perceptual processing. We consider this in more detail in our next section where we examine how these key elements of mathematics and scientific context combine to create not only meaning, but reality itself in the mind of the perceiver in order that we may bring these components to the table in the right combination and sequence to ignite the synergistic integration in the mind of the student.

Science Combined with Mathematics Create Reality

Barrett (2017) explained in detail the importance, subtlety, and nature of the perceptual processing that gives a hint of our goal of weaving sensory, symbolic, diagrammatic, and literary representations together into the fabric of reality autonomously created as we experience the world around us:

> Without concepts, you'd experience a world of ever-fluctuating noise. Everything you ever encountered would be like everything else. You'd be experientially blind, like when you first saw the blobby picture in chapter 2, but permanently so. You'd be incapable of learning[5].
>
> All sensory information is a massive, constantly changing puzzle for our brain to solve. The objects you see, the sounds you hear, the odors you smell, the touches you feel, the flavors you taste, and the interoceptive sensations you experience as aches and pains and affect … they all involve continuous sensory input signals that are highly variable and ambiguous as they reach your brain. Your brain's job is predict them before they arrive, fill in missing details, and find regularities where possible, so you experience a world of objects, people, music, and events, not the "blooming, buzzing, confusion" that is really out there[6].
>
> To achieve this magnificent feat, your brain employs concepts to make the sensory signals meaningful, creating an explanation for where they came from, what they refer to in the world, and how to act on them. Your perceptions are so vivid and immediate that they compel you to believe that you experience the world *as it is*, when you actually experience a world *of your own construction*. Much of what you actually experience as the outside world begins in your head. When you categorize using concepts, you go beyond the information available, just as you did when you perceived a bee within blobs. (pp. 85-86)

> So what's happening in your brain when you categorize? You are not finding similarities in the world, but *creating* them. When your brain needs a concept, it constructs one on the fly, mixing and matching from a population of instances from your past experience, to best fit your goals in a particular situation. (p. 92)

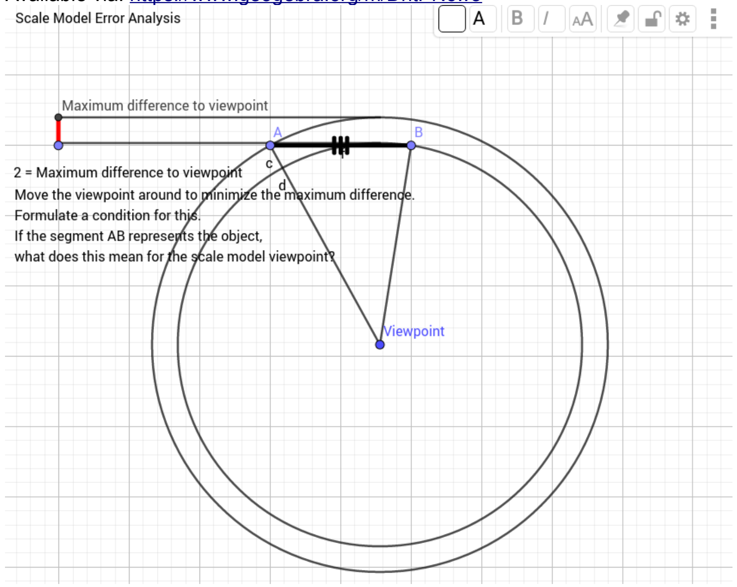
As compelling and as rich with implication as this is for integrating science and mathematics in education, there is a specific aspect that has not yet been adequately acknowledged. The crucial influence of linking representational diagrams, symbols, concepts, and associated meaning together with words as Barrett (2017) explained:

> Some concepts are learned without words, but words confer the distinct advantages to a developing conceptual system. A word might begin as a mere stream of sounds to the infant, just one part of the whole statistical learning package, but it quickly becomes an invitation for the infant to *create* similarities among diverse instances. … (p. 98)

> Words encourage infants to form goal-based concepts by inspiring them to represent things as equivalent. In fact, studies show that infants can more easily learn a goal-based concept, given a word, than a concept defined by similarity without a word.[31](p. 98)

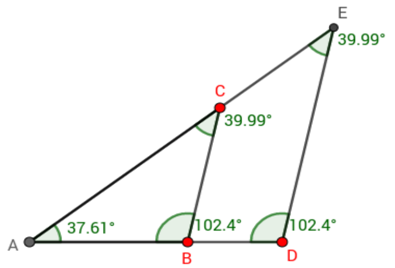
> Words encourage infants to search for similarities beyond the physical, similarities that act like a mental glue for concepts. … (p. 100)

Scale Model Building Error Analysis GeoGebra Activity
Available via: https://www.geogebra.org/m/DhtPNcw3



Scale Model Building GeoGebra Similar Triangle Activities 1 and 2
Available via: https://www.geogebra.org/m/ptRQrB3P

Similar triangles are triangles that have corrsponding angles equal.
In the diagram below triangles ABC and ADE are similar.
We write this symbolically as: $\triangle ABC \sim \triangle ADF$



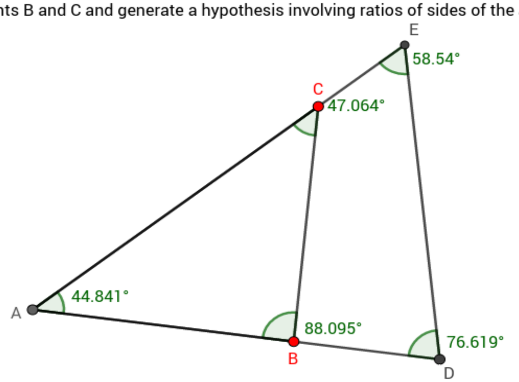
BC/AC = 0.62
BC/AB = 0.95
DE/AE = 0.62
DE/AD = 0.95
DE/BC = 1.6
AE/AC = 1.6
AD/AB = 1.6

Drag points B, C, and D around to notice the relationships.
Generate a hypothesis you think is true for corresponding parts of similar triangles.
Useful words: angle correspond ratio shape side size.

Corresponding parts: parts that are similar, associated, or linked in character, meaning, or function. Example: Side b corrsponds to side d. Angle C corresponds to angle E.

Ratio: A relation between two numbers that forms a fraction or rational number.
Example: The ratio of students to chairs is 16 to 24, 16/24, or about 0.66.

Available via: https://www.geogebra.org/m/Bn2Nz4Yg
Similar Triangles Part 2:
When are triangles that have a common angle at A similar?
Drag points B and C and generate a hypothesis involving ratios of sides of the angle at A.



BC/AC = 0.706
BC/AB = 0.963

DE/AD = 0.827
DE/BC = 1.338

AD/AB = 1.559
AE/AC = 1.303

Thus the importance of linking together the various associations of diagrams, symbols, patterns represented in each, and corresponding real world-entities together categorically with words. However, words not only bring life to these distinct aspects as a common concepts, they preload our perceptual machinery with these corresponding categories using the disparate facets together with their associations as contextual cues. Above and beyond all of this cognitive arrangement, words also allow the transcendence beyond the inner self, by bringing them into the realm of interpersonal communication described so aptly by Barrett (2017):

> Conceptual combination is powerful, but it is far less efficient than having a word. If you asked me what I had for dinner this evening, I could say “baked dough with tomato sauce and cheese,” but this is much less efficient than than saying “pizza.” … If you want the concept to be efficient, and you want to transmit the concept to others, then a word is pretty handy.
> Infants can benefit from this “pizza effect” before they can speak. For example, prelinguistic infants generally can hold about three objects in mind at a time. If you hide toys in a box while an infant watches, she can remember up to three hiding places. However, if you label several toys with nonsense words like “dax” and several more with “blicket” before hiding them – assigning the toys to categories – the infant can hold up to six objects in mind! This happens eve if all six toys are physically identical, strongly suggesting that infants gain the same efficiency benefits from conceptual knowledge that adults do. *Conceptual combinations plus words equals the power to create reality* [emphasis added]. (pp. 105-106)

O’Halloran (2005) described one aspect of the extrapolation common for the higher levels of mathematical and scientific conceptualization resulting from the synergistic association of symbols, diagrams, networks of connections, meaning, and categories together with names and words that brings to life the scientific world view:

> Language, symbolism, and visual images function together in mathematical discourse to create a semantic circuit which permits semantic expansions beyond that conceivable through the individual contributions. The resultant meaning potential of mathematics therefore stretches beyond that possible through the sum of the three resources. Following this view, the success of mathematics as a discourse stems from the fact that it draws upon the meaning potentials of language, visual images, and the symbolism in very specific ways. That is, the discourse, grammar, and display systems for each resource have evolved to function as interlocking system networks rather than isolated phenomena. (p. 159)

When we consider the full depth of this integrated combination of words, symbols, visuals, and grounded, meaningful sensory experiences of underlying patterns in the real world, the full picture starts to emerge. Let’s bring it to life with this name: a **full featured scientific model**. This is exactly where establishing deep connections between various representations and patterns in the real world can have the most dramatic effects. This is the crucial inherent synergy we should target early and deeply with integrated mathematics and science education. Embedding mathematics within science changes our perceptions, of not only mathematics and science, but the real world and our abilities to understand and affect it.

Kellman, Massey, and Son (2009) described and explain the targeting of autonomous perceptual processing within education as what has become known as perceptual learning (PL):

> Discovery pertains to finding the features or relations relevant to learning some classification, whereas fluency refers to extracting information more quickly and automatically with practice. Both discovery and and fluency differences between experts and novices have since been

found to be crucial to expertise in a variety of domains such as science problem solving … , and mathematics … (p. 287)

Eleanore Gibson, who pioneered the field of PL (for a review, see Gibson 1969), defined it as "an increase in the ability to extract information from the environment, as a result of experience ..." (p. 3). She described a number of particular ways in which information extraction improves, including both the discovery and fluency effects noted above. Of particular interest to Gibson was "… discovery of invariant properties which are in correspondence with physical variables" (Gibson 1969, p. 81). This view of PL applies directly to many real-world learning problems; although Gibson did not mention mathematics learning, her examples included … (p. 287)

… It is common to consider PL as described by Fahle and Poggio (2002): "… parts of the learning process that are independent from conscious forms of learning and involve structural and/or functional changes in the primary sensory cortices."

They study the effects of three perceptual learning module (PLM) interventions for middle and high school students on various representations of linear functions to conclude (Kellman, Massey, & Son, 2009):

The study of PL interventions in education and training has barely begun, yet the promise is already clear. PL techniques have the potential to address crucial, neglected dimensions of learning. These include selectivity and fluency in extracting information, discovering important relations, and mapping structure across representations. Each PLM described here addressed an area of mathematics learning known to be problematic for many students. In each case, a relatively short intervention produced major and lasting gains, and in each case the learning transferred to key mathematical tasks that differed from the training task. (p. 301)

Integrated mathematics and science education can be most effective when extending these perceptual learning modules into sequences of lessons centered around developing and helping found the various perceptual facets of full-featured scientific models, being explicit about the roles of science, mathematics, literacy, and technology in order to showcase resulting deeper understanding and ability to effect positive and lasting change in the world.

Mathematical and Scientific Literacy

Mathematical and scientific literacy captures the ability to perceive, reason with, and expound, both verbally and in written form, on the models we use in a profound way to make sense of the world around us. It is at the core of both mathematics and science education. Indeed, Pearson, Moje, and Greenleaf (2010) noted "Scientific literacy has been the rallying cry for science education reform for the last 20 years …" further specifying the science inquiry viewpoint of scientific literacy that:

… makes explicit connections among the language of science, how science concepts are rendered in various text forms, and resulting scientific knowledge (5). Researchers guided by this latter view are concerned with how students develop the proficiencies needed to engage in scientific inquiry, including how to read, write and reason with the language, texts, and dispositions of science. The ability to make meaning of oral and written language representations is central to the robust science knowledge and full participation in public discourse about science. (p. 459)

Wellington and Osbourne described it this way (2001):

Cole: Perspective for Artists

glass (Fig. 4), is also determined by these rays, and that the near one looks longer than the distant one.

We have seen that the height and width of objects as they appear to us is determined by the converging rays from their extremities to our eye; that objects really equal in size appear shorter and narrower when further away.

Depths of objects on a level surface traced.—It only remains to find out that the depths on a receding surface are governed by the same laws.

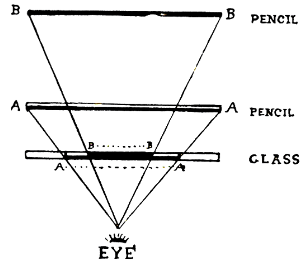


FIG. 4.—Two pencils, glass, and eye, as seen from above (i.e. ground plan).

Fig. 5 represents three pinheads in a row, one behind the other, on the far side of the glass from the position of the eye. Notice, however, that the eye is above the pins (i.e. looking down on them),

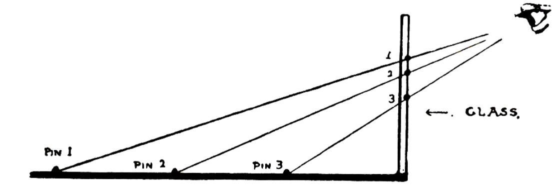


FIG. 5.—Side view (i.e. elevation) of the painter's eye, an upright glass, and a level board on which three pins are equally spaced.

and so the points where their rays cut the glass are one above the other in regular order, the nearest pin (3) appearing the lowest down on the glass.

Since the pins were placed at equal distances apart, their spacing, as shown on the glass, would also fix the depths of the ground surface between them (Fig. 6).

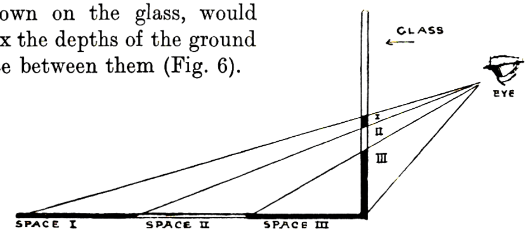


FIG. 6.—Same as Fig. 5, showing the spaces between the pins and as they appear on the glass.

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Tracing on glass.—If we look at an object through a sheet of glass we can trace on that glass the apparent height or width of that object (Fig. 2); in other words, we can mark off on the glass those

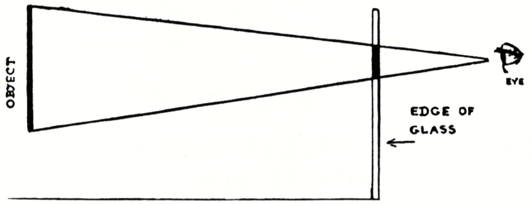


Fig. 2.—Upright sheet of glass, object, and eye; showing the rays from the extremities of the object passing through the glass and marking its height on it.

points where the rays from the extremities of the object on the way to our eye pass through it.

Height of objects at varying distances traced on glass.—If we now place two objects of similar height one behind the other (Fig. 3) our tracing of each discovers the one farthest off to appear on the glass shorter than the one close at hand. Fig. 3 makes it evident

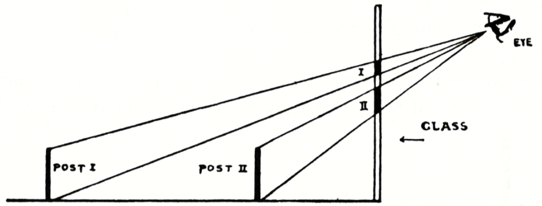


Fig. 3.—Side view (i.e. elevation) of posts, an upright glass, and painter's eye.

that this apparent difference in size is due to the fact that the converging rays from the further object have the longest distance to travel, and so are nearly together where they pass through the glass. On the other hand, the rays from the object close to the glass have only just started on their journey and so are still wide apart.

Width of objects traced.—Let us repeat the experiment with two pencils of equal length lying on a flat surface, one behind the other. We shall be satisfied that their apparent length, as traced on the

> Learning science is, in many ways, like learning a new language. … But learning to use the language of science is fundamental to the learning of science. As Vygotsky (1962) pointed out, when a child uses words he or she is helped to learn concepts. Language development and conceptual development are inextricably linked. Thought requires language, language requires thought. (pp. 5-6)

When we consider the multifaceted language of diagrams, symbols, and words that are used to represent patterns, relationships, and concepts fused into a full-featured scientific model, we can see that a form of education that embraces the complexity of form, activity, emphasis, and engagement is required. Wellington and Osbourne came to this conclusion about their roles as teacher of science (2001):

> We are, primarily, raconteurs of science, knowledge intermediaries between the scientific canon and its new acolytes. Such an emphasis means that we must give prominence to the means and modes of representing scientific ideas, and explicitly to the teaching of how to read, write, and how to talk science. … (pp. 138-139)

Finally, O'Halloran (2005) described the linkage between scientific and mathematical literacy and, when we consider the points brought forth in the previous pages, argued implicitly for the integration of mathematics and science education:

> Mathematical and scientific language involve particular types of linguistic choices which organize reality in particular ways. Mathematics and science are registers where particular configurations of experiential, logical, interpersonal, and textual meanings are found. … Mathematical and scientific language developed in particulary ways because the dynamic construal of reality was delegated to mathematical symbolism where the relations could be displayed visually. (p. 200)
>
> Finally, to reiterate an important point, mathematical and scientific language cannot be viewed in isolation. The nature of selections and the lexicographic strategies for encoding meaning needs to be seen in relation to the mathematical symbolism and visual display. (p. 201)
>
> In an educational context, students are typically presented with modern mathematics in a pre-packaged form where the functions and grammar of mathematical symbolism are not discussed. The discourse is presented in such a way that many students do not understand what mathematics is, or how mathematical symbolism developed historically as a semiotic resource in order to fulfill certain functions. Given that the grammar of mathematical symbolism is not taught from a linguistic perspective, the cumulative effect is that many students fail mathematics because they simply do not understand how (or why) mathematical symbolism functions as a resource for meaning. (p. 202)

However, we can begin to address these issues in a way that makes learning fun and teaching inspiring, by exploiting crucial inherent synergies by embedding mathematics in scientific contexts, being explicit about and extending content that is being used, and starting early.

Start Early

Starting early in middle school fosters favorable dispositions towards science and mathematics that not only increase student motivation, but change the nature of the students relation with further STEM study and careers at the crucial time when attitudes are developing (Redmond et al., 2011, p. 399), yielding higher confidence, value of content, and effort put forth for both mathematics and science (Redmond et al., 2011, p. 403).

Newcombe (2010) highlighted the importance of spatial thinking in STEM, gives evidence to support its malleability, and suggested enriching activities for primary grades that overlap directly, naturally, and are extended by integrating mathematics and science: reading challenging content, imagining and visualizing trajectories, working with spatial puzzles, using spatial language, using maps and models, and developing analogies (Newcombe, 2010, p.34).

Kellman, Moje, and Son summarize their linear measurement PLM results noting that it “produced significant improvement in the sixth grade intervention group” and furthermore that they “achieved nearly identical scores on a delayed posttest administered 4.5 months later, indicating that their learning gains were fully maintained” (Kellman, Moje, & Son, 2009, p. 300). Again, early changes in perception produce lasting effect in understanding and cannot help but change attitudes.

Integrating mathematics and science education may not replicate these techniques in their intensity, or entirety, but can to some extent allow a more inherent activation of these “crucial, neglected dimensions of learning“ that “include selectivity and fluency in extracting information, discovering important relations, and mapping structure across representations.” (Kellman, Moje, & Son, 2009, p. 301).

Starting early may also help avoid the negative perception that “teachers, administrators, and parents worry whether students are really learning or simply playing” when “students greatly enjoy integrated lessons” without any directly visible and easily quantifiable gains (Redmond et al., 2011, p.400). I suppose this too, in analogy with mathematical symbolism, is a matter of educational perceptual learning based in discovery and fluency.

Build Syntactic Form Dictionary Entries

Building entries in the dictionary of relational meanings, accessible through various indices including symbolic and experiential patterns, a key to fluent literacy and a major goal of integrated mathematics and science education. Fundamental entries in this dictionary may be best identified with the basic symbolic forms described by Sherin (2001). In this article, Sherin presented “a semi-exhaustive list of symbolic forms, along with a number of brief examples” (Sherin, 2001, p. 505) clustered by similarity meaning or function. For example, his “competing terms cluster” contains symbolic forms with parts that contribute or compete at the same level or in a similar way (Sherin, 2001, p. 506):

Symbolic Form	***Symbol Pattern***
Competing Terms	□ ± □ ± □ …
Opposition	□ - □
Balancing	□ = □
Canceling	0 = □ - □

These symbolic forms are fundamental to the application of mathematical symbolism to patterns. Elementary students are expected to internalize many of these fundamental these before secondary education begins. However, beyond these basic symbolic forms are fixed points of conceptualization I’ll call **symbolic idioms**. Symbolic idioms associate deeper embedded meaning involving multiple

Scale Model Building Reading 2
Cole (1921/1976) *Perspective For Artists*
pp. 17-19.

Cole: Persepective For Artists

PART I

NATURE’S PERSPECTIVE AS SEEN AND USED DAILY BY PAINTERS

CHAPTER I

THE PRINCIPLE OF PERSPECTIVE IN THEORY

“ If you do not rest on the good foundation of nature, you will labour with little honour and less profit.”—Leonardo da Vinci.

LINEAR Perspective is a study that deals with the appearance of objects[1] as regards their size and the direction of their lines seen at varying distances and from any point of view. When practising it we are not concerned with their apparent changes of colour or tone, though those also help us to recognise the distance separating us, or that of one object from another.

Visual rays.—The Theory of Perspective is based on the fact that from every point of an object that we are looking at, a ray of light

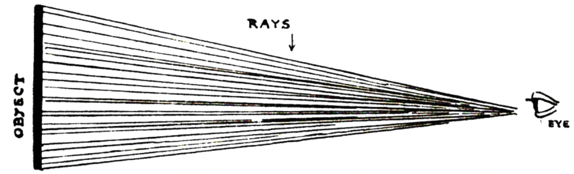


Fig. 1.

is carried in a straight line to our eye.[2] By these innumerable rays we gain the impression of that object (Fig. 1).

[1] “ Objects ” is a mean word to use, because perspective laws also apply to the surface of the earth, the sea, and the sky, and all living things. It is used for convenience.

[2] We see objects at a different angle according to whether we have both eyes open, the left shut, or the right shut. When drawing objects very close at hand look with one eye only.

Taking some basic facts seriously, scrutinizing them and developing them to extremes is the way science generally develops. The tools for this development are sophisticated instrumentation for finding experimental facts and mathematical theory for refining the ideas; what seems like magic is a brew of applied mathematics.

But before going into mathematical detail, we can already deduce some aspects of the geometry of light by thinking about things we already know, encouraged by the saying that "research is to see what everybody has seen and to think what nobody has thought" (Jammer [1989]). We know, for example, that a convex lens focuses light (Fig. 1.2); parallel bundles of light rays are focused at one point, which suggests that in the geometry of light established by the lens parallel lines meet. The Greek mathematician Euclid, who developed geometry from five axioms, postulated that parallels never meet, but Euclid's geometry is the geometry of flat space. Euclid's parallel axiom is in fact the defining characteristic of flat space. The light rays focused by the lens do not seem to conform to Euclid's postulate; the geometry of light is non–Euclidean, light may perceive a medium as a curved space. Only in exceptional cases is the geometry established by an optical material that of flat space. ...

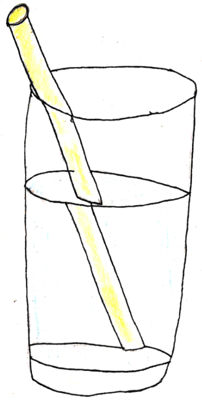


Figure 1.1: Refraction. The image of a straw in a glass of water appears refracted at the water surface. (Credit: Maria Leonhardt.)

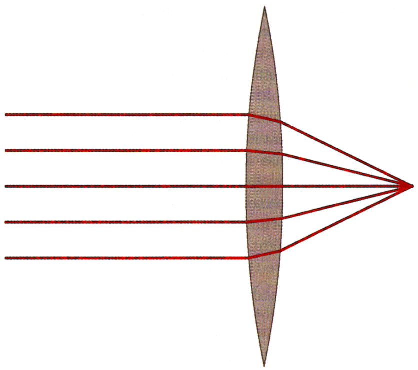


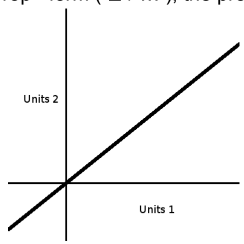
Figure 1.2: Parallel light rays (red) meet in the focus of a lens (grey).

symbolic forms, concepts, and/or visualizations that are implied or hidden beneath a shallow interpretation of the symbols. For instance, Sherin touches on the deeper meaning associated with identity form (x = …), but, as described in the next paragraph below, it is has multiple meanings so fundamentally important that it is not just a symbolic form, but belongs in the category of symbolic idiom. Furthermore, it is most accessible in the context of a full featured scientific model where many times its use is so natural as to be "nearly invisible" (Sherin, 2001, p. 518). The identity form combines multiple meanings (object vs process stance – described below) all within the one symbolic form. However, we'll also take a look at other symbolic idioms that are formed from an interlocking combination of symbolic forms yielding a multi-faceted concept of simultaneous meanings.

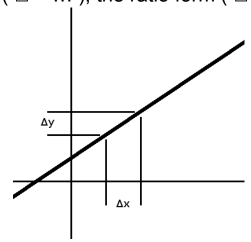
Four main symbolic idioms form a core of ideas that often feature strongly in middle school algebra, and form a strong base for associating meaning and symbols: the identity idiom, the direct proportion idiom, the rate idiom, and the equal proportions idiom. Furthermore, the last three idioms are not only interrelated and applied individually throughout science and mathematics, but can also sometimes be transformed from one to another to gain a different perspective and added meaning.

The *identity idiom* (x = …), the use of the equals sign in the object stance [S p.518], is one of the most significant in mathematics and science. Especially when the left hand side, for readers of left-to-right languages, is a single variable or quantity. It encourages categorization through specification that brings to life useful concepts. When that quantity becomes part of a crucial understanding, we often end up associating a name with it, bringing it to life more qualitatively. We see this aspect of the idiom in the definition of variables within traditional mathematics problem solutions. At the same time, it gains meaning in combination with its process stance that describes a computational structure within the definition of the right hand side. For example, consider the following words and associated identity idioms: force (f = ma), work (w = fd), and voltage (V=IR). In each case the left-hand-side defining the quantity in terms of others, and at the same time describing one way to process quantities to get it. Furthermore, once the meaning of the left hand side is fully founded, it extends beyond either of these immediate meanings (object and process) where it flows through the equality to the right hand side via implied operations that shift the focus to some other symbol (for example: m = f/a, f = w/d, R = V/I). The context of use reinforces these various facets of meaning, and it becomes obvious that symbols without context, or patterns without diagrams and symbols, are missing crucial components.

The *direct proportion idiom*, is a combination of the balancing form (□ = □), ratio form (□ / □), the prop+ form (□ / …), the prop- form (… / □), and the scaling form (n □). It even has its own special symbol that combines the balancing form and scaling form (□ ∝ □, eg., a ∝ b, read as "a is directly proportional to b"). The underlying concept is the equivalence of measures of a common entity using two different, proportional units and with their ratio forming a conversion factor. A prototypical example is the equivalence of 25.4 millimeters and 1 inch. Each symbolic form carries deeper associated meanings of equivalence and other symbolic forms: balancing 25.4 mm = 1 in, ratio 25.4 mm / 1 in, ratio 1 in / 25.4 mm, scaling with these, and the role of specific measurement in proportion numerator and denominator. This should also spark a connection with "multiplying by 1" and the concept of unchanged values. Also to be associated with this are images of 2D line graphs through the origin relating the two measures. Note, however, that this idiom only applies to directly proportional units, where a measurement of 0 in one system gives a measure of 0 in the other, unlike degrees Celsius and Fahrenheit. This idiom is also associated with the rate idiom.

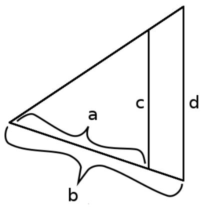


The *rate idiom*, a prime middle school development goal, is a special combination of the identity form (□ = …), the ratio form (□ / □), the change form (Δ = □ - □), and the base ± change form (□ ± Δ).



A prototypical example is the slope of a line, especially when perceived in the form ($m = \Delta y / \Delta x$). Here the identity form should invoke, to varying extents: the object and process stances, the ratio form, the change forms, an associated a 2D line graph, and a perceptual context such as temperature. Perhaps, this might also cause some activation of the base ± change form in the sense of $y_2 = m\,\Delta x + y_1$. However, this idiom is foundational, especially when combined with the identity idiom for the definitions of velocity, acceleration, and other similar quantities that link changes through proportion. This idiom is one of the cornerstones at the foundation of calculus. It is hard to imagine someone comprehending calculus and its applications without this landmark symbolic idiom.

The *equal proportions idiom*, is a geometric variation of the rate idiom that is seen in a couple of main symbolic forms. The first form is a specialized notation developed specifically for this idiom and



carries over into the literary realm with the symbolic template intact (□ : □ :: □ : □**,** eg., a:b::c:d, read as "**a** is to **b** as **c** is to **d**"). A slight symbolic variant uses the a combination of the balancing and ratio forms (□ / □ = □ / □) in one of a number of ways that vary according to which values occupy the numerators and which the denominators. However, the underlying idiom should activate to some extent multiple balanced ratios. The situation is often visualized through the geometric diagram of similar triangles and associated with the corresponding symbolic forms: a:b::c:d, a/b = c/d, b/a = d/c, as well as the alternative forms a:c::b:d, a/c = b/d, b/a = d/c. The underlying relation should also activate to some secondary extent a related balance form: ad = bc. Again, until this idiom is deeply embedded and perceptually available with only the slightest of cues, it is better when coupled with some meaningful context such as will become available in the scale model building section below.

The equal proportions idiom is more fundamental than the rate and proportional equivalence idioms in the sense that many of our algebraic notions are grounded in geometric perceptions as we can see in these visualizations. It is worthwhile perceptually for students to be able to visualize similar triangles in each of the these other visualizations (even though the horizontal and vertical lines are absent in the direct proportion graphic, they can be imagined).

Scale Model Building

Scale model building is one of a choice of projects suitable for early integration of science and mathematics at the beginning of middle school in a setting that includes explicit science and mathematics where the content is not necessarily (from the point of view of students) the primary goal of the lessons. One possible configuration focused to develop a meaningful, useful, and full featured scientific model and symbolic idiom is described in detail in this section. Other popular choices for early integration might include:

- Model rocketry with a focus on rocket trajectory for proper limitation of height and proper deployment of a parachute.
- Simple machines (inclined plane, levers and fulcrum, wheel and axle, rope and pulley, gears) with a focus on lifting a heavy block using a limited torque motor.

Scale Model Building Reading 1
Leonhardt & Philbin (2010) *Geometry and Light : The Science of Invisibility*
pp. 1-2. to mid-3rd paragraph + 2 diagrams.

Chapter 1 | Leonhardt & Philbin *Geometry and Light : The Science of Invisibility*

Prologue

Many mass–produced products of modern technology would have appeared completely magical two hundred years ago. Mobile phones and computers are obvious examples, but something as commonplace to us as electric light would perhaps be just as astonishing to an age of candles and oil lamps. It seems reasonable to assume that we are no more prescient than the children of the Enlightenment and that, as science and technology develop further, some things that appear impossible today will become ubiquitous in the future. As Arthur C. Clarke famously wrote, "Any sufficiently advanced technology is indistinguishable from magic". In this book we focus on optics and electromagnetism, an ancient subject so suffused with notions of magic that the word illusion is still used by its modern practitioners in their learned journals. We explain the science of the ultimate optical illusion, invisibility. The ingredients of invisibility can be used for other surprising optical effects, such as perfect imaging and laboratory analogues of black holes. Just as important as the particular applications discussed are the powerful ideas that underlie them, ideas that have a fascinating pedigree and that are far from exhausted. We hope to equip the reader with these versatile and fruitful tools of physics and mathematics.

Although invisibility may seem like magic, its roots are familiar to everyone with (literal) vision. Almost all we need to do is to wonder and ask questions. Take a simple observation from daily life and ask some questions: if a straw is placed in a glass of water it appears to be broken at the water's surface (Fig. 1.1). We know the straw is not really broken (and miraculously repaired when removed from the water), so what does the water change? It can only change our perception of the straw, its image carried by light. The water in the glass distorts our perception of space, and this perception is conveyed by light. We conclude that the water changes the measure of space for light, the way light "sees" distances—the geometry of space. Other transparent substances like glass or air, called optical materials or optical media, should not be qualitatively different from water in the way they distort geometry for light. So we are led to the hypothesis that media appear to light as geometries. In this book we take this geometrical perspective on light in media seriously and develop it to extremes. We also discuss its limitations and find the conditions when the geometry established by media is exact.

1

Materials

Some of the lesson activities listed above include include formative assessments (*Think-Pair-Share, Concept Card Mapping*) that are more fully described in Keeley & Tobey's marvelous book of formative assessments (Keely & Tobey, 2011). A variety of directed reading and expository activities and techniques are described in Wellington and Osbourne's book of science literacy (Wellington & Osbourne, 2001).

The readings and GeoGebra worksheets that follow were selected (and developed) to elucidate the intent of the lesson outlines and this thesis as a whole. They, like the lesson outlines above, are untested on actual sixth grade students and will need the oversight of a magical educator along with additional support and/or modification to be effective in helping achieving the educational standards and reaching the goals outlined here. Once the lessons work well, detailed lesson plans and associated materials can be compiled, replicated, and packaged to be used for observation and training or used directly.

Choice is as important as the project setting for the tone of engagement. The project should be something students want and choose to do, and must be the focus of the lessons. This allows science and mathematics to be introduced in their natural settings, as tools for understanding and prediction of quantitative value to accomplish the real-world project. Ideally, students would select projects based on interests and intentions, much the same way college students plan out a college degree: with some requirements and some freedom of choice.

Scale model building has the potential to develop intuitions and skills in common problematic areas:

- Development and use of a full featured scientific model of light and vision, complete with diagrams, words, and symbols in well grounded and easily accessible fundamental perceptions forming landmark symbolic forms and symbolic idioms.
- Development, practice, and integral use of measurements with full grounding in measurement tools and associated symbolic forms.
- Practice visualizing spatial relationships including similar triangles and symbolizing the associated direct proportion idiom.
- Meaning making with fractions and the associated ratio and proportion symbolic forms.
- Meaning making through directed reading activities of text through associated diagrams, perceptions, and symbols.
- Meaning generation and expression through writing that includes words, diagrams, and symbols.

Scale model building supports development of a full featured scientific model that combines the following ideas to generate model part sizes from photo measurements in a way that (roughly) preserves the proportions of the parts of the original object:

- For non-luminous objects, light impinges on objects where it is either reflected or absorbed and re-emitted.
- Light travels in straight lines (important tie-in here for future science).
- Objects are visible when the light coming from them makes its way into our eyes.
- Direct proportion idiom (as described above) including associated symbolic statements: a:b::c:d, a:c::b:d.
- A photograph can be modeled as lying in a plane perpendicular to the light rays from the original object to an eye/camera.
- A scale model can be modeled as lying in a plane perpendicular to the light rays from the original object to an eye/camera.
- Some sort of unifying diagram that includes similar triangles within i) object plane, ii) photo plane, iii) perpendicular to the eye, photo, and object planes.
- Qualitative analysis of model applicability and breakdown.
- Narrative to explain and summarize the full featured scientific model including diagrams, symbolic statements, example calculations, and words to tie them all together and relate them to real world entities and relationships.

Primary goals of the project are to develop a full featured scientific model to understand how visual images represent objects using the mathematics of scalability, to use the model to generate (predict) values, to use the values to create the scale model, and, finally, to explain the science and mathematics with diagrams, symbols, and words in a poster-board presentation. Beyond these abstract goals we want to engage students to cross-link and apply these various conceptualizations through basic kinesthetic and visual perceptions, intent, and action by having each student (or group of students): select an appropriate photo, measure various parts of the photo, construct and assemble model pieces, and embellish the scale model (to strengthen ties to real world objects). These lessons should have the feel of an active, shop-like class, with a showcase finale.

The outline of lesson plans that follow break the project down into 12 lessons of 90 minutes each. The first two lessons introduce science and mathematics content with associated activities ending with some brainstorming for the full featured scientific model. The third lesson pulls together an appropriate diagram, symbolic formulas, and explanations into the full featured scientific model of scale model building from photographs for the project. Construction can begin in the 4th lesson for those with adequate notes. These six lessons (4th through 9th) are for calculations and construction of models. Projects should start wrapping up in the 10th lesson (including any painting and decorations), with 15 minutes reserved to review the use of science, math, diagrams, symbols, and explanations ahead of the 11th lesson reserved for presentation development. One final lesson for presentation and gallery as part of the school community that includes parents and guests.

In the lesson outlines that follow, some general associated Next Generation Science Standards (NGSS) with the three main subcategories Disciplinary Core Ideas (DCI), Science and Engineering Practices (SEP), and Crosscutting Concepts (CCC) are listed with the direct science instruction. Similarly, general associated Common Core State Standards for Mathematics (CCSSM) are listed with the direct mathematics instruction. Again, these are rough outlines only meant to suggest intention. No doubt, various CCSS Literature standards are addressed are addressed within as well, most notably through the notebooks and final presentation.

Lesson 0: Introduction of project choices.

- Overview of scale model building with example scale model:
  - Display a scale model from previous projects.
    Mention science of light and vision, perspective drawing, and geometry of similar triangles.
    Mention development of scientific model incorporating these.
    Mention use of photos, rulers, wood, saws, glue and decoration.
    Mention writeup.
  - Video clip: http://vimeo.com/139407849 (Overstreet & Gorosh, 2015)
    Did the scale model change your idea of the solar system? How?
    What did the builders do to build the model? (measurements, make and arrange parts)
    What is the same about the model and the solar system? What is different? (shape)
    What might scale models be useful for? (visualize, understand, explain)

- Other project choices (not described further, but perhaps: model rocketry, simple machines)

Lesson 1: Science of light and vision, readings, and discussion.

- ***Light and perception reading and activity:***
  Leonhard & Philbin *Geometry and Light : The Science of Invisibility*
  pp. 1-2. to mid-3rd paragraph + 2 diagrams (see attachments).
  - Activity of straw in water:
    Move the straw from side-to-side in the glass.
    Write down observations and explanations.
    What changes as the straw moves from one side to the other? Why?
  - Teaching option: design an experiment.
  - Think, Pair, Share:
    Where does the light come from? (sun, lights, fire, … energy)

simultaneously to establish and embed the equal proportions idiom and associate its various facets with the idiomatic entry in the symbolic dictionary of meaning. It also helps students practice the critical skill of visualizing the geometry and corresponding symbology. This idiom is used not only repeatedly in the full featured scientific model for the project, but will also appear again and again in further science and mathematics. Ideally, the project embeds this idiom deeply into the dictionary of each student so that related future cues easily activate the associated visualizations, symbolic forms, and words to not only generate meaning*ful* perceptions, but furthermore facilitate the “natural and comfortable use of mathematics” (Redish & Kuo, 2015) “with understanding” (Sherin 2001) that is the hallmark of scientific and mathematical literacy.

Secondary project goals include take away lessons that are often separated out into segregated science and mathematics classes as well as the practice of beneficial and/or problematic skills:

- familiarization with some science of light and vision that will become increasingly important with the increasing role of robotics and automated processing,
- familiarization with similar triangles and the associated equal proportions,
- familiarization with perspective drawing,
- practice measuring,
- practice using fractions,
- and practice visualizing relationships and representing them symbolically.

Another unique and beneficial result of the project lessons as outlined above is the qualitative investigation of the error/limitation of the model. This lends an engineering aspect of the project that is ripe for differentiation or further study in mathematics and art.

As important as the scientific literacy, and science and mathematics content, are the attitudes fostered by the project. Beyond the use of mathematics and science to achieve the project, the final project presentation and gallery has a crucial role to play in the educational scheme due to the potential impact of feelings engendered. Feelings of accomplishment will add to the resilience of students as well as to the value of mathematics, science, and education for them. Feelings of pride and recognition will encourage the continuation in science, technology, engineering, and mathematics related endeavors. Feelings of leadership and representation will encourage participation within the systems of education and society. Observation by primary students will set expectations of achievement within and enjoyment of the educational process as well as set the stage for proper appreciation of mathematics, science, and engineering.

Conclusion

Science, with its roots in observation and its goal of explaining the patterns of the world around us, provides an ideal context for grounding and giving value to mathematics. Mathematics, as a compact visual, symbolic, and literal language of patterns and relations, has evolved with science to represent and reason with these embedded patterns, sometimes taking its own trajectories leaving the world behind. Making worthwhile meaning with patterns is the link that binds them together. Mathematics is the language designed to tell the amazing stories of science, that loose something in translation and told in any other language. To be able to use both science with mathematics to first comprehend the understood and then explain the unknown must be demonstrated and practiced with students in order for them to learn the language and the stories so that they can succeed in our increasingly complex and interconnected world. Scale model building provides an ideal first step on this road in its potential to develop a small, but crucial and fully interlinked piece of meaning: the equal proportions symbolic idiom. Furthermore, it has the potential deeply imprint this idiom with an enjoyable, interesting, meaningful, and useful story, while at the same time showcasing some of our finest tools of the trade of comprehension.

- the scientific observations of geometry of light and perspective, directly as well as previewed in the readings,
- the concrete association of equal proportions not only with visualizations of the corresponding similar triangles of the GeoGebra activities, both passively and actively through manipulations of the GeoGebra worksheets, but also with the fractions of the physical objects, both real-world, and various stages of the scale model,
- the aggregation of the culminating representations of diagrams, symbols, and words both in notebooks and on poster-board,
- and finally, the physical manifestation of the project, that, hopefully, along with a variety of other triggers and contextual cues, will refresh the representations and associations each time it comes to mind.

As mentioned in the prior list, kinesthetics also help ground the content and entire experience with the following activities, each of which has active and perceptual components:

- drawing perspective representations on glass with grease pencils,
- manipulating GeoGebra worksheets,
- measuring photographs,
- converting photograph measurements to scale model measurements,
- moving about the room to acquire modeling materials and use tools,
- measuring out scale model parts,
- cutting scale model parts,
- assembling the scale model from the scale model parts,
- writing in notebooks,
- generating and assembling poster-board presentations.

One of the key components of the project, in addition to the development and use of both science and mathematics content, is the intentional incorporation and explicit description of the use of each content area in the project: the explicit discussion of the role of science in the observation, analysis, and modeling of phenomena, as well as the role of mathematics in providing the concepts and a compact language of quantitative patterns. This not only helps explain why we teach these content areas and what their roles are in the wider world, but also helps form the attitudes toward using these as tools to understand, decide, and take effect. One aspect that will only become visible in follow-on projects is the transference of knowledge through familiarity underlying patterns and context specific knowledge.

Just as with the content roles, explicit emphasis is placed on the use of various roles and the resulting benefits of the forms of representation: diagrams, symbols, and words. This is combined with the implicit emphasis of actually using these representations in the various activities of the project. This can even be extended to a 4th representation form, virtual modeling or computer simulation, exemplified in the GeoGebra worksheets. Notice that the lessons attempt to associate symbols with diagrams with phrases to help consolidate different facets of a concept. Of particular importance and explicit emphasis is the theme that just as words help represent and facilitate thought, the synergistic combination of diagrams, symbols, and words help represent and facilitate mathematical and scientific thought as made explicit in the full featured scientific model of the project. The incorporation of these representational components within the context of the project provide visibility of their roles and benefits that would otherwise be absent.

Specifically, notice the multiple activations of the diagrams, symbols, and name of similar triangles in close association with activations of representations for corresponding equations of equal proportions in the context of using these various representations themselves to represent real world objects (and trajectories). This lays the foundation for grounding these representations activating them all

How does light interact with the: straw, water, air?
(material classes: opaque, transparent, translucent)
Did the image of the straw in water look weird? Why? (distance through water changes)
Would it look weird if it, and we, were immersed in water?
(no: homogeneous medium and adjustable perception, yes: depth perception differs)
How are light, geometry, and our perceptions linked?
(opaque objects reflect light through media to eyes, geometry for paths: flat=lines)

- ○ Write: What is geometry and how does it play a part in this? (above into notebooks)
- ○ Two Sided Debate: When we look at a building, is the geometry flat or not?
  What about people with binoculars/glasses?
  What about the lens in your eye?
  What about water in the air?
- ○ Discuss: How does the combination of words and diagram help us understand?
- ○ Discuss: What are pros and cons of modeling light and vision with flat geometry?
- ○ Discuss: How does noticing and investigating weird behavior lead to scientific discovery?
- ○ NGSS SEP Analyzing and interpreting data.
  NGSS SEP Constructing explanations for science.
  NGSS CCC Cause and and effect: Mechanism and explanation.
  NGSS CCC Structure and function.
  NGSS DCI PS4.B Electromagnetic radiation: light reflected, absorbed, transmitted.
  Moves in straight lines that depend on the media.

- ***Perspective reading and activity:***
  Cole *Perspective For Artists* pp. 17-19 (see attachments).
  - ○ Activity of tracing on glass (room corner patterns)
    Need glass pane, stand, and grease pencils for each student.
    Need to set up classroom with patterns for students to trace on glass for worksheet.
    (Need depths of such length that give ratios that differ from height and width.)
  - ○ Worksheet to identify effects (described in reading) with portion of tracing by color.
    - ▪ Trace and compare same height at different distances. 2 sets.
    - ▪ Trace and compare same width at same distances. 2 sets.
    - ▪ Trace and compare same depth starting at same distances. 2 sets.
    - ▪ Compare ratios of heights, widths, and depths.
  - ○ Discuss: How does distance affect sizes?
    How does angle of view affect what we see?
  - ○ NGSS SEP Analyzing and interpreting data.
    NGSS SEP Constructing explanations for science.
    NGSS CCC Cause and and effect: Mechanism and explanation.
    NGSS CCC Structure and function.
    NGSS DCI LS1.D Information processing: receptors respond to electromagnetic radiation.
    Signals are then processed in the brain.
- ***Assignments***:
  - ○ Watch: Ken Dill at TEDxSBU (show at lunch or after school)
    The Protein Folding Problem: A Major Conundrum of Science
    https://www.youtube.com/watch?v=zm-3kovWpNQ (Dill, 2013)
  - ○ Research: Investigate interesting objects to build.

Lesson 2: Similar triangles GeoGebra activities, brainstorm application to vision.

- GeoGebra activities for **similar triangles** (see attachments)
  - Explore and generate hypotheses (need to get and emphasize these):
    - Similar triangles have adjacent sides of corresponding angle in the same proportion (ΔABC ~ ΔADE → AB/AD = AC/AE, as well as = BC/DE).
    - Triangles with the same angle and corresponding sides in **equal proportion** are similar. (AB/AD = AC/AE → ΔABC ~ ΔADE).
  - Introduce special symbolic notations:
    - a:b::c:d means a/b=c/d as well as b/a=d/c.
    - ΔABC ~ ΔADE means ΔABC is similar to (has the same shape as) ΔADE (order matters: corresponding sides and angles).
  - Think-Pair-Share ratios that stayed the same: discover & name the **scale factor** (AB/AD).
  - Note/discuss the association of diagrams, symbols, and words for the associated concepts of **similar triangles**, **equal proportions**, and **scale factor**.
  - Compare historic hypotheses with statements and proof.
    (Late in Euclid: proportions and ratios are complicated, no symbols.)
    - Display English translation of 2000 year old Greek text.
      Heath *Euclid, The Thirteen Books of the Elements, Volume II, Bookx III-IX.*
      pp. 194-204. Propositions VI.2, VI.5, and VI.6 statements and diagrams only.
    - Note/discuss the use of diagrams, words, and symbols and how they correspond.
    - Note/discuss the statements of proposition only (no proofs).
    - Map student hypotheses to closest Euclid proposition.
    - Read through Venema's preface section about proof:
      Exploring Advanced Euclidean Geometry with GeoGebra (Venema, 2013 p. ix).
  - Teaching option: Rational numbers as a number system beyond Greek math.
    (Deep connection here in density difference of reals=segments and rationals=fractions.)
  - CCSSM 6.RP Understand ratio concepts and use ratio reasoning to solve problems.
    CSSM 6.EE Apply and extend previous understandings of of arithmetic to algebraic expressions.

- Brainstorm use of similar triangles, photographs, and measurements
  - Concept Card Mapping: Groups generate a concept network from cards:
    - Light moves through air in straight lines.
    - Distant objects are smaller, closer objects are larger.
    - A combination of words and diagrams can help explain.
    - Triangle in object plane represents object.
    - Triangle in photo plane represents photo.
    - Triangle from eye to object.
    - Triangle from eye to photo.
    - Diagram of similar triangles and proportion of scale factor.
    - Diagram of perspective concept.
  - Students generate sketches that include, object, photo, eye=camera.
    Good student sketch on document camera (if possible).
    See lesson 3 for goal diagram(s) and associated similar triangle use.
    - What should be in the drawing?
      (object, eye=viewpoint, lines of light for key points, photograph, labels of points)
    - How can that help figure out scale modeling?
      (similar triangles, convert photo measurements to scale)

- Poster-board presentations should include:
  - Diagrams for representing: lines of sight, near plane, far plane, eye, labels.
  - Photograph with key measurement and scale factor to convert to scale model measure.
  - Symbols for similar triangles, equal proportions, equal ratios, conversion equation with an independent and dependent variable and identification of a scale factor.
  - Two example calculations one of which verifies the scale factor.
  - Written description of how light, geometry (of light), and perspective generate a model of scalings.
  - Diagram and written description of inherent error and/or limitation of the model as used for the project.
  - Written descriptions of how science, mathematics, or models help us understand or effect the world.
  - Personal reflection on the project and personal meaning that should include difficulties and epiphanies.

Lessons 12: Presentations, model gallery, and celebration.

- Models and poster-boards on display.
- Other students (including elementary students as possible), teachers, parents, guests invited browse and ask questions.

Discussion

These lessons are structured to accomplish the primary goals of engaging students with pieces of science, mathematics, and context in a focused way that allows formulation of a full featured scientific model utilizing diagrams, symbols, words and, most importantly, internal interconnections within content among these various representations as well as connections to perceivable real-world representations that help ground representations and the experience. If we now consider the various components of synergy mentioned before the lesson outlines, the ideal nature of the scale model building project comes into view.

The beginning is key. The introduction of the various projects puts the focus on the projects rather than science and mathematics content. Offering a choice of projects activates engagement of students and sets the foundation of attitudes wherein science and mathematics are used to help accomplish a desired goal. Moreover, the science and mathematics for scale model building are interrelated so that each content area brings value of deeper understanding to the other, and integrates synergistically in the full featured scientific model to help understand and accomplish the project. This makes explicit the interlinked nature of science in discovering how things work and corresponding mathematics that helps formulate usable expressions of relational patterns.

Next, notice how the primary sense of vision grounds all aspects of the experience. Students see:

- (optionally, hopefully) the real-world object,
- the photograph of the object, both perceptively and actively (through intentional component selection and measurement, and hopefully, the taking and printing of the photograph),
- the lines of sight and light both diagrammatically and physically (via the perspective activity),
- the diagram linking the crucial components of the model and the experience, including multiple instantiations of the symbolic idiom of equal proportions in the context of visualizing similar triangles, both at the crucial model formulation stage, in notes, and at the final presentation,
- the symbolic sentences of similarity and proportion, strongly associated with both visualizing diagrams and real world objects,

        - Perpendicular to the center of the the object.
        - A far distance from the object.
        - Differentiate by adding technical error analysis?

- Group Quiz: Example calculation.
    - Suppose a scale model of a bridge with a span of 1 meter is desired, and the span of the bridge in the photograph measures 25 centimeters. The hight of the bridge in the photo is 15 centimeters. How tall with the bridge be in the scale model?
    - Examine various student solutions, reasoning, and checks and emphasize:
        - Rearrangement of the calculation from $\frac{EF}{BC} * BD$ to $\frac{BD}{BC} * EF$.
        - That BD/BC is the fraction of the model that BD is.
        - That when multiplied by EF gives the corresponding fraction of the model EG.
        - Measurement units matter, and that like units in numerator and denominator reduce leaving the proper (scale model) units.
          (Work in dimensional analysis for differentiation?)
        - Check scale factor: replacing BD with BC should give EF.
- Finalize notes in notebooks. Research and determine real world objects and photographs.
  (No one will be allowed to proceed to construction without adequate notes!)
    - Student notes: Full featured scientific model (diagram, symbolic statements, words).
    - Student notes: Error analysis (diagram, words).
    - Student notes: Scale factor calculation (if possible).
      Maximum model size 1 meter x 1 meter x 1 meter?
    - Student notes: Plan for construction (materials, order of work, ...)
    - Make available: stock photographs of bridges and buildings for models.
- Assignment: Find an object to build and associated photograph(s).

Lessons 4-9: Construct scale model.

- Teacher reviews and approves notebooks.
- Teacher facilitates review of scientific model as appropriate.
- Teacher facilitates construction as appropriate including:
    - Rules for classroom construction (safety with tools).
    - Advice (Measure twice, cut once. Staged development. Use symmetry.)
    - Demonstrations of use (ruler, tape measure, mitre box, handsaws, hot glue)
- Teacher helps discuss special topics:
    - Fraction manipulation (have student explain their calculations)
    - Dimensional analysis (examples: height, speed, acceleration)
- Students calculate, note, build.

Lessons 10,11: Finish models and notebooks. Generate poster-board presentation.

- Key summary discussions to develop:
    - What is science and how does it help us? (models that explain/predict how things work)
    - What is math and how does it help us? (reason/specify quantitative patterns)
    - How do models help us? (identify important parts and how they interact)
    - What are scale models? Scale factor? (models that preserve size and shape)
    - What other (than size & shape) models might be useful?

    - Get it all in notebooks including.
      Diagram including object, photo, eye=camera.
      Explanation of how to use photo.
      Explanation of why it works, scale factor, types of error.
    - NGSS SEP Developing and using models.
      NGSS SEP Using mathematical and computational thinking.
      NGSS SEP Constructing explanations for science.
      NGSS CCC Similar triangles patterns in nature.
      NGSS CCC Scale and proportion relationship.
      NGSS DCI ETS1.B Developing possible solutions.
    - CCSSM 6.RP Understand ratio concepts and use ratio reasoning to solve problems.
      CCSSM 6.EE Apply and extend previous understandings of of arithmetic to algebraic expressions.
      CCSSM 7.G Draw, construct, and describe geometrical figures and describe the relationships between them.

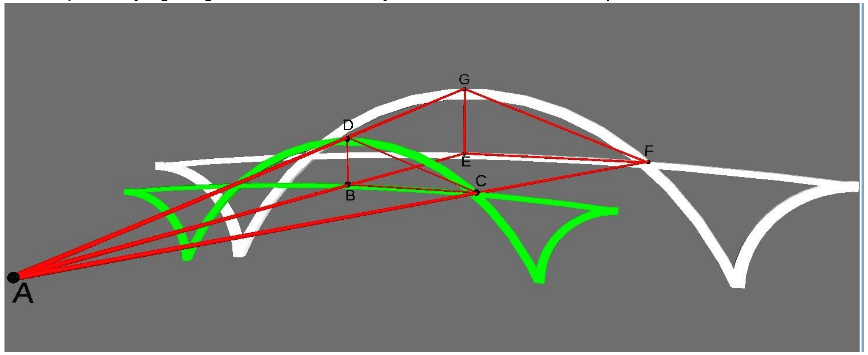
- Explain scale model construction materials and and methods:
    - Each model should have one scale factor per photo.
    - Display materials (balsa sheets, beams, dowels, and plywood base).
    - Bridges and buildings have proven workable in previous lessons.
    - Size limit (30 inches x 30 inches x 30 inches).
    - Display tools (nippers, hand-saws, mitre boxes, rulers, compasses, tape measures, glues).
    - Recall types of photos needed (possibly multiple views).
    - Work in groups or individually, but everyone needs their own photos and notebook.
    - Poster board presentation at end: diagram, explanations, photos and scale factors, example calculations.
    - Need adequate notes to proceed to construction.
    - Stress safety, responsibility, and behavior due to dangers of materials and tools.
      (Once construction starts, safety goggles required!)

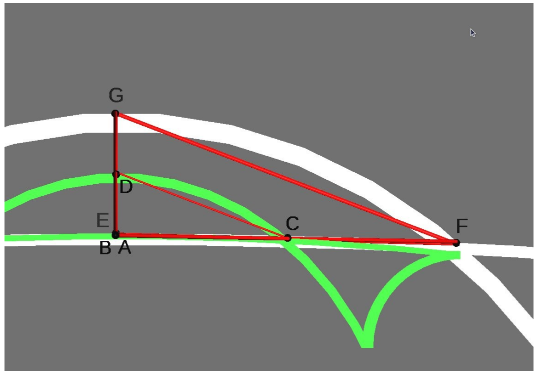
- Students update notebooks:
    - Similar triangles: shape, ratios, scale factor, symbols.
    - Diagram for project.
    - Ideas about models.

- Assignment:
    - Consider how all these pieces fit together to help us build build a scale model from photographs.
    - Identify an object to build.
    - Watch: Tim's Vermeer (show during lunches or after school)
      http://www.sonyclassics.com/timsvermeer/ (Jillette & Teller, 2013)

Lesson 3: Review scientific model, review subject and photo selections, start calculations.

- Develop a unifying diagrams, associated symbolic formulas, and explanations:





  - Use similar triangles within and across near and far planes for the following points.
    When the objects in the near and far planes have the same orientation to A
    (the two planes are parallel):
    - ΔABD ~ ΔAEG → AB : AE :: BD : EG → AB/AE = BD/EG
    - ΔABC ~ ΔAEF → AB : AE :: BC : EF → AB/AE = BC/EF
    - BD/EG = AB/AE = BC/EF and
    - angle B = angle E → ΔBCD ~ ΔEFG
    - Use words: Same angle at B and E with corresponding sides in equal proportions means triangle BCD is similar to triangle EFG.
    - The ratio BC/EF (= BD/EG = CD/FG) is the **scale factor** from ΔEFG to ΔBCD
      Multiply by EF we get BC.
      The ratio EF/BC (= EG/BD = FG/CD) is the **scale factor** from ΔBCD to ΔEFG
      Multiply by BC we get EF. Multiply by BD to get EG. Multiply by CD to get FG.
      We use the same **scale factor** for all 3!
      See how the symbols give quantitative relations!

    - Notice how different aspects are all associated and captured in the symbols: **equal proportions**, **scale factor**, and **similar triangles**.
    - Notice that science of light and vision is used to formulate our model and diagrams.
      The diagrams identify and relate parts of the model, science explains how they work.
    - Notice that mathematical symbols make patterns and reasoning explicit compactly.
      The diagrams and symbols fit on a page, and can be in our minds simultaneously.
      The symbols encode patterns in the diagram.
      We see the pattern of the symbols to know that BC/EF (or …) is the scale factor.
      Mathematics helps us formulate and reason with quantitative patterns.
    - Notice that our model is bound together into the phrase **scale factor** as a specific concept that becomes a short hand for all of what we know about scaling.
    - Notice that our combination of science and mathematics, represented with diagrams, symbols, and words forms an understanding that far exceeds the results of any one of science, mathematics, diagrams, symbols, or words alone. This **full-featured scientific model** is the core of scientific and mathematical literacy fundamental to our modern scientific world view.
    - Case 1: Near plane ↔ photograph, far plane ↔ real-world object.
    - Case 2: Near plane ↔ photograph, far plane ↔ scale model (when model > photo).
      Then we need to multiply photo measurements like BC by the scale factor EF/BC to get
      model sizes like EF. We get EF when we decide how big the scale model will be!
      Take BC as the largest measurement of the object (span of a bridge) in the photograph,
      and the corresponding size EF as the overall size of the model.
      Introduce measurement variables:
      p for photograph measurment
      m for corresponding model measurement
      Write an equation that relates them using the scale factor: p * BC/EF = m
      Various equivalent forms, but this one has a procedural flavor.
    - Case 3: Near plane ↔ scale model, far plane ↔ real-world object.
    - Teaching option: similar triangle transitivity and corresponding scale factors.
    - Teaching option: proof of uniformity of similarity.
    - Teaching option: anamorphism error analysis. (This one suggested and included next!)
  - CCSSM 6.RP Understand ratio concepts and use ratio reasoning to solve problems.
    CCSSM 6.EE Reason about and solve one variable equations and inequalities.
    CCSSM 6.EE Represent and analyze quantitative relationships between dependent and independent variables.
    CCSSM 7.G Draw, construct, and describe geometrical figures and describe the relationships between them.
  - NGSS SEP Developing and using models.
    NGSS SEP Using mathematical and computational thinking.
    NGSS SEP Constructing explanations for science.
    NGSS CCC Similar triangles patterns in nature.
    NGSS CCC Scale and proportion relationship.
    NGSS DCI ETS1.B Developing possible solutions.

- Anamorphism Error Analysis
  - Commit and Toss: A 300m bridge has rail posts spaced every 10m. Will the posts be evenly spaced in a photograph of the bridge?
  - GeoGebra activity (see attachment)
  - Conclusion: To minimize error, the photograph should be taken so that the viewpoint is: